SEQUENTIAL LIFE TESTING

by

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INTRODUCTION

The purpose of this paper is to show some of the aspects of sequential life testing. This area of statistics is relatively new. Most of the work has been done since 1953.

The major contributor to sequential life testing has been Benjamin Epstein, of Wayne University. Epstein has had the following papers published: "Truncated life tests in the exponential case," (1954), "Simple estimators of the parameters of exponential distributions when samples are censored," (1956), "Exponential distribution and its role in life testing," (1958), "Tests for valididty of the assumption that the underlying distribution of life is exponential," (1960a), "Statistical life test acceptance procedures," (1960b), and "Estimation from life test data," (1960c). Epstein also has done some work with Milton Sobel of the University of Minnesota. Together they have had the following papers published: "Life testing," (1953), "Some theorems relevant to life testing from an exponential distribution," (1954), and "Sequential life tests in the exponential case," (1955).

Life testing is used where the data which we are concerned with is the length of life of the product, machinery or anything which discontinues because of failure. There are several types of failure. An item can fail through normal constant wear, fail only when a force is exerted on the item, or fail because of an accident which damages or destroys the item.

The first chapter of this report deals with the underlying distribution of sequential life testing and some related theorems. Both the pros and cons of using the exponential distribution are given. Chapter II is concerned with hypothesis testing. This is usually in terms of the mean life of the

population under study. The usual hypothesis to be tested is

$$H_o: \theta = \theta_o \text{ vs}$$

$$H_1: \theta = \theta_1 < \theta_o .$$

In Chapter III, various procedures are given for estimating the parameters such as the mean life, the expected time, T, it will take until a decision is made, and the expected number of items which will fail, r, before a decision is reached.

CHAPTER I

UNDERLYING DISTRIBUTIONS

Most of the work which has been done in the field of sequential life testing, has been done on the assumption that life situations follow the exponential distribution. This distribution is sometimes referred to as a random failure pattern.

From Epstein (1958), let t = life time of an item.

$$E(t) = \int_o^\infty t \; f\,(t;\theta)\; dt \tag{1}$$

$$= \int_o^\infty \frac{t}{\theta} \exp\left(\frac{-t}{\theta}\right) dt$$

$$= \theta \quad .$$

A random failure pattern means that the cause of failure occurs according to a Poisson process with rate λ. The distribution of time between failures is given by the probability density function (p.d.f.)

$$\lambda e^{-\lambda t} \qquad \lambda > 0,\ t > 0 \ . \tag{2}$$

Let the random variable (r.v.) T be defined by

$$Pr(T > t) = Pr(\text{no failure occurs between 0 and } t)$$

Now assuming the failure process is Poisson, the following can be shown.

$$Pr(T > t) = e^{-\lambda t} \tag{3}$$

$$Pr(T \leq t) = 1 - e^{-\lambda t} \quad \text{whose p.d.f. is} \tag{4}$$

$$f(t) = \lambda e^{-\lambda t} . \tag{5}$$

The above is used when the item can fail only at certain times when forces are placed on it. So then it is desirable to know the probability of failure, given a force has been exerted.

Define

p = probability of a failure given a force has been exerted.

$$0 \leq p \leq 1$$

$$q = 1 - p .$$

Now we have the situation where a force has been exerted once on the item and the item hasn't failed, the force has been exerted twice and the item hasn't failed, etc. Symbolically, this is given by

$$Pr(T > t) = e^{-\lambda t} + q(\lambda t)\, e^{-\lambda t} + q^2 \frac{(\lambda t)^2}{2!}\, e^{-\lambda t} + \ldots + q^k \frac{(\lambda t)^k}{k!}\, e^{-\lambda t} + \ldots$$

$$= e^{-\lambda t}\left[1 + q\lambda t + \frac{(q\lambda t)^2}{2!} + \ldots + \frac{(q\lambda t)^k}{k!} + \ldots\right]$$

$$= e^{(-\lambda t + q\lambda t)}$$

$$= e^{-\lambda p t} \quad . \tag{6}$$

Therefore

$$\Pr(T \leq t) = 1 - e^{-\lambda pt} \tag{7}$$

with a p.d.f.

$$f(t) = \lambda p \, e^{-\lambda pt} \, . \tag{8}$$

All of Epstein and Sobel's work has been done on the assumption that time until failure follows the exponential distribution. Let's now look at some theorems (Epstein and Sobel, 1954) which are relevant to life testing under the exponential distribution.

The distribution of the life length of any single item is assumed to have the following density

$$p(x;\, \theta,\, A) = \begin{cases} \frac{1}{\theta} e^{-(x - A)/\theta} & \text{for all } x \geq A \\ 0 & \text{otherwise} \quad . \end{cases} \tag{9}$$

Suppose N items are divided into k sets S_j with each set containing $n_j > 0$ items. Each set is observed only until there are r_j failures

$$0 < r_j \leq n_j \quad .$$

There are three different cases which we can consider.

1. The n_j items in each set have a common known A_j ($j = 1,2,\ldots,k$).
2. All N items have the same unknown A.
3. The n_j items in each set have a common unknown A_j ($j = 1,2,\ldots,k$).

A is a constant, and can be interpreted two different ways.

1. A is the minimum life, which is taken as zero.
2. A is the "time of birth," life is measured from time A.

Let

$$R = \sum_{j=1}^{k} r_j \tag{10}$$

where

$$X_1 \leq X_2 \leq \ldots \leq X_{r_j}, \qquad 1 \leq r_j \leq n_j$$

denotes the r samllest ordered observations. This set of n_j r.v.'s represents a typical set S_j.

The following properties can be shown to be true (Epstein and Sobel, 1954). Let

$$V_j = \sum_{i=1}^{r_j} (X_{ji} - A_j) + (n_j - r_j)(X_{r_j} - A_j) . \tag{11}$$

Then $2V_j/\theta$ is distributed as a chi-square with $2r_j$ degrees of freedom $[\chi^2(2r_j)]$. Let

$$V_j' = \sum_{i=1}^{r_j} (X_{ji} - X_{1j}) + (n_j - r_j)(X_{r_j} - X_{1j}) . \tag{12}$$

Then $2V_j'/\theta$ is distributed as $\chi^2\ (2r_j - 2)$. Let

$$V_j^* = \sum_{i=1}^{r_j} (X_{ji} - c) + (n_j - r_j)(X_{r_j} - c) . \tag{13}$$

Then $2V_j^*/\theta$ given $X_1 \geq c$ is distributed as $\chi^2\ (2r_j)$.

The maximum likelihood estimator of θ for cases 1, 2, and 3 respectively, is given as follows:

$$\hat{\theta}_1 = \sum_{j=1}^{k} V_j/R, \quad \hat{\theta}_2 = \sum_{j=1}^{k} V_j^*/R, \quad \text{and} \quad \hat{\theta}_3 = \sum_{j=1}^{k} V_j'/R .$$

Theorem 1. The distribution of $\hat{\theta}$ depends only on R, θ, and, in case 3, also on k. The random variable $2R\hat{\theta}/\theta$ is distributed as $\chi^2(2R)$, $\chi^2(2R-2)$, and $\chi^2(2R-2k)$ in cases 1, 2, and 3 respectively.

The following properties are given by Epstein and Sobel, (1954);

$$T_1 = \sum_{j=1}^{k} \left[\sum_{i=1}^{r_j} X_{ji} + (n_j - r_j)\, X_{jr_j} \right]$$

$$= R\hat{\theta} + \sum_{j=1}^{k} n_j A_j \tag{14}$$

$$T_2 = (T_{2o}, T_{21}) \tag{15}$$

$$T_{2o} = T_1$$

$$T_{21} = \min_j X_{j1}$$

$$T_3 = (T_{3o}, T_{31}, \ldots, T_{3k}) \tag{16}$$

$$T_{3o} = T_1$$

$$T_{3j} = X_{j1} \qquad j = 1, 2, \ldots, k \ .$$

Observations through T_i (i = 1, 2, 3) are all that is needed to estimate θ_i.

Theorem 2. T_1 is sufficient for estimating θ .

Theorem 3. $T_2 = (T_{20}, T_{21})$ is sufficient and complete for estimating the pair (θ, A).

Theorem 4. T_3 is sufficient and complete for estimating (θ, A, k).

While Epstein and Sobel have been using the assumption that the exponential distribution is the one which should be used in life testing,

others have shown that the exponential distribution only follows if the failure rate is termed constant. If the failure rate is non-constant, then the Weibull distribution or the gamma distribution (Tate, 1959) tend to fit the failure data more closely.

Zelen and Dannemiller (1961) stated,

> We have tried to show that dogmatic use of life testing procedures without a careful verification of the assumption that failure times follow the exponential distribution may result in a high probability of accepting "poor quality" equipment.

It has been shown (Zelen and Dannemiller, 1961) that when the Weibull distribution is used instead of the exponential, the estimate of the mean life does not have to be as high for the Weibull, in order to get the same say 90 or 95% lower limit. In other words, if one wishes to make a statement that at least 95% of the items will live for a certain length of time, the mean life under the Weibull distribution does not have to be as high as it would be for the exponential distribution.

Further attacks on the use of the exponential as the underlying distribution for life situations were made by Birnbaum and Saunders (1958). They developed a model which is a compromise with the exponential, Weibull and gamma distributions. They stated,

> The usefulness of the exponential distribution is sharply limited due to the following property: one can prove that if the life length T of a structure has the exponential distribution, then previous use does not affect its future life length.

For some things such as jeweled bearings in watches, the previous use does not seem to have any affect on the failure at a given time. But most functional items become weaker or more susceptible to failure the longer they are in use.

If the damage or wear on an item is defined as δ, and the effect of δ on the item being tested, at time t, is

$\delta(t)$.

One can reasonably assume that δ should have an effect on the distribution of the life length. Let's write the cumulative distribution function (c.d.f.) and the p.d.f. as

$$\Pr(T \leq t|\delta) = F(t;\delta) \quad \text{for all } t \geq 0 \tag{17}$$

$$\frac{dF(t;\delta)}{dt} = f(t;\delta) \quad \text{for all } t \geq 0 \; . \tag{18}$$

The failure rate at time t is now defined as

$$\gamma_\delta(t) = \frac{f(t;\delta)}{1-F(t;\delta)} \; . \tag{19}$$

Epstein (1960a) wrote an article defending the use of the exponential distribution. If a distribution is exponential and the c.d.f. is

$$F(t) = 0 \qquad t < 0 \tag{20}$$

$$F(t) = 1 - e^{-t/\theta} \quad t \geq 0,\ \theta > 0 \tag{21}$$

define

$$y = \log\left(\frac{1}{1-F(t)}\right)$$

$$= t/\theta \tag{22}$$

and if we plot y against t, we get a straight line with slope $1/\theta$. This procedure gives a graphical technique for testing the validity of the exponential.

Another method for testing the validity of the exponential distribution is by dividing the horizontal axis on a graph into k intervals. Observe

the number of failures in each interval, then calculate the expected number of failures for each interval

$$e_i = np_i \quad .$$

Then find

$$\chi^2 = \sum_{i=1}^{k} \frac{(0_i - e_i)^2}{e_i} \qquad (23)$$

and compare this result with $\chi^2(\alpha,k-1)$, where $\chi^2(\alpha,k-1)$ is the value exceeded α% of the time by a $\chi^2(k-1)$ r.v. If the $\chi^2 < \chi^2\ (\alpha,k-1)$, then there is no reason to reject the exponential as the underlying distribution.

Lomax (1954) looks at a different type of failure than the ones which we have been discussing; this is the failure of a business. In general it can be said that the longer a business is able to survive, the smaller the probability of a failure becomes. The data given below is on business failures in Poughkeepsie from 1844-1926.

Correlation Coefficients for Functions
Fitted to Data on Conditional Probability

Type of Business	Exponential	Hyperbola
Retail	0.91	0.99
Manufacture	0.96	0.83
Craft	0.93	0.99
Service	0.91	0.98

The function

$$\frac{c}{t} \qquad c > 0,\ t > 0$$

for the hyperbola is easier for calculating purposes than the function

$$\frac{1}{\theta} e^{-t/\theta}$$

for the exponential.

Much of the early work in life testing was done on electron tubes. There are two classifications of failures in electron tubes; catastrophic or sudden failures, and wear-out or delayed failures. It was from this work with electron tubes that the exponential distribution was found to best fit the failure distribution.

Even though there has been much discussion on the validity of the exponential distribution throughout the rest of this report, the time till failure will be assumed to be distributed exponentially, unless otherwise stated.

CHAPTER II

HYPOTHESIS TESTING

One of the main reasons for performing a sequential life test is to test the hypothesis

$$H_o: \theta = \theta_o \text{ vs}$$

$$H_1: \theta = \theta_1 < \theta_o \quad .$$

These hypotheses apply, for example, if a company has developed a new product and wants to know if the mean life is at least a certain length of time.

The first type of problem which will be discussed is one that was explained by Epstein and Sobel (1953). From a given population, n items are drawn at random. When there is a failure of one item, it is not replaced. In the future, this will be called the nonreplacement case. The life test

will be terminated at which ever comes first of two preassigned values r, a certain number of failures, $r \leq n$, or T_o, a given length of time. This will be denoted by

$$\min (r, T_o) .$$

If there are r failures before time T_o, then the null hypothesis is rejected. If time T_o occurs before r failures, then H_o is accepted.

The above procedure was derived from the following; test

$$H_o: \theta = \theta_o \text{ vs}$$

$$H_1: \theta = \theta_1 < \theta_o .$$

Accept H_o if $\hat{\theta}_{r,n} > C$ and reject H_o if $\hat{\theta}_{r,n} < C$, where

$$\hat{\theta}_{r,n} = \frac{x_{1,n} + x_{2,n} + \ldots + x_{r,n} + (n-r)\, x_{r,n}}{r} \tag{24}$$

and $x_{j,n}$, $j = 1, 2, \ldots, r$ represent the life length of the jth failure.

The next question which arises is, "How do we find r and C?"

If we are given θ_1 and θ_o we can find r and C for chosen values of α and β. These values for r and C may be found from tables by Eisenhart (1947). First we form the ratio θ_o/θ_1 and then choose the values from the table for r and C which correspond to this ratio and our chosen values for α and β.

$$L(\theta_o) = \Pr(\text{accept } \theta = \theta_o \text{ given } \theta_o \text{ true}) = 1 - \alpha \tag{25}$$

$$L(\theta_1) = \Pr(\text{accept } \theta = \theta_o \text{ given } \theta_1 \text{ true}) \leq \beta . \tag{26}$$

What would happen if we would use only the last item which failed? Since there is so much emphasis given to $x_{r,n}$, it is only reasonable to assume

that this value is worth more than the previous values. It can be shown that we can choose n large enough, using the same r, so that our power for accepting H_o if

$$x_{r,n} > C_1$$

is almost identical to the curve based on

$$\hat{\theta}_{r,n} > C_o \quad .$$

Let's look at an example of this situation. Let

$$\theta_o = 1500, \quad \theta_1 = 500, \quad \alpha = \beta = .05$$

$$\frac{\theta_o}{\theta_1} = 3 \ .$$

From the table, (Eisenhart, 1947) we find that $r = 10$. Using $r = 10$, the power curve is nearly as good using $x_{r,n}$, as it is using $\hat{\theta}_{r,n}$, as long as $n \geq 14$. In this particular example, $C = 540$. Suppose that a sample of size $n = 20$ is taken. We now look at $x_{10,20}$, and if

$$x_{10,20} > 540$$

we accept H_o, and if

$$x_{10,20} \leq 540$$

we reject H_o. The situation which we discussed earlier is now evident. If any

$$x_{j,n} > 540 \qquad j = 1, 2, \ldots, r-1,$$

we can stop the experiment and accept H_o .

The expected values of r and T are given by Epstein (1954) .

If the probability of an item failing before T is given by

$$p = 1 - e^{T_o/\theta} \tag{27}$$

then from the binomial law, we find that the probability of making a decision with exactly k failures is

$$\Pr(r = k|\theta) = bn(k;n,p_o)$$

$$= \binom{n}{k} p_o^k (1 - p_o)^{n-k} \quad k = 0, 1, \ldots, r_o-1 \tag{28}$$

$$\Pr(r = r_o|\theta) = 1 - \sum_{k=0}^{r_o-1} bn(k;n,p_o) \tag{29}$$

$$E(r) = \sum_{k=0}^{r_o} k \Pr(r = k|\theta) \; . \tag{30}$$

From the above statement, we can simplify to a form which is easier to calculate,

$$E(r) = np_o \left[\sum_{k=0}^{r_o-1} bn(k;n-1, p_o) \right] + r_o \left[1 - \sum_{k=0}^{r_o-1} bn(k;n,p_o) \right] . \tag{31}$$

Now by using binomial tables or tables of the incomplete beta function, we can determine the expected value of r for any given n, T and r. Let's show that

$$E(T) = \sum_{k=1}^{r_o} \Pr(r = k|\theta)\, E(x_{k,n}) \; . \tag{32}$$

The proof is as follows

$$E(T) = T_o \left[\sum_{k=0}^{r_o-1} bn(k;n,p) \right] + \sum_{k=r_o}^{n} bn(k;n,p)\, E(x_{r_o,n}|r=k) \tag{33}$$

$$E(x_{r_o,n}) = \sum_{k=0}^{r_o-1} bn(k;n,p)\, E(x_{r_o,n}|r=k)$$

$$+ \sum_{k=r_o}^{n} bn(k;n,p)\ E(x_{r_o,n} | r = k) \tag{34}$$

$$E(T) = E(x_{r_o,n}) + \sum_{k=0}^{r_o-1} bn(k;n,p) \left[T_o - E(x_{r_o,n} | r=k) \right] . \tag{35}$$

From (32), we get the following result

$$E(x_{r_o,n} | r=k) = T_o + E(x_{r_o-k,n-k})\quad k = 1, 2, \ldots, r_o-1 . \tag{36}$$

since $E(x_{r-k,n-k})$ is the unconditional expected waiting time to get the (r-k)th failure in a random sample of size n-k. It has been shown (Epstein, 1954) that for

$$1 \leq k \leq n$$

$$E(x_{k,n}) = \theta\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{n-k+1}\right) . \tag{37}$$

Therefore

$$E(x_{r_o-k,n-k}) = E(x_{r_o,n}) - E(x_{k,n}) \quad 1 \leq k \leq r_o \tag{38}$$

$$E(x_{r_o,n} | r=k) = T_o + E(x_{r_o,n}) - E(x_{k,n}) \tag{39}$$

$$E(T) = E(x_{r_o,n}) + \sum_{k=0}^{r_o-1} bn(k;n,p_o) \left[-E(x_{k,n}) + E(x_{r,n}) \right]$$

$$= E(x_{r_o,n}) - \sum_{k=0}^{r_o-1} bn(k;n,p_o) \left[E(x_{r,n}) + E(x_{k,n}) \right] . \tag{40}$$

Now we will consider the case where an item fails and it is replaced by a new one (replacement case). Now we will always have n items on test.

The same procedure as before will be followed; stop the experiment at $\min(x_{r_o,n}, T_o)$. The replacing of failed items now makes the life test a Poisson process with failures occurring at the rate

$$\lambda_o = \frac{n}{\theta} \tag{41}$$

The probability of reaching a decision in exactly k failures now becomes

$$\Pr(r = k|\theta) = p(k;\, \lambda_o,\, T_o)$$

$$= \frac{1}{k!} e^{-nT_o/\theta} (nT_o/\theta)^k \tag{42}$$

$$k = 0, 1, 2, \ldots, r_o - 1$$

$$\Pr(r = r_o|\theta) = 1 - \sum_{k=0}^{r_o-1} p(k;\, \lambda_o,\, T_o) \tag{43}$$

$$E(r) = \sum_{k=1}^{r_o} k \left[\frac{1}{k!} e^{-nT_o/\theta} (nT_o/\theta)^k\right]$$

$$= \sum_{k=1}^{r_o} \left[\frac{1}{(k-1)!} e^{-nT_o/\theta} (nT_o/\theta)^{k-1}\right] \lambda_o T_o$$

$$= \sum_{k=1}^{r_o} k \Pr(r = k|\theta) \,. \tag{44}$$

Again we simplify for the purpose of calculations

$$E(r) = \lambda_o T_o \left[\sum_{k=0}^{r_o-2} p(k;\, \lambda_o,\, T_o\right] + r_o \left[r \sum_{k=0}^{r_o-1} p(k;\, \lambda_o, T_o)\right]. \tag{45}$$

From any preassigned values of n, T and r, we can find E(r) from Molina's tables of the Poisson distribution (1949).

The proof of

$$E(T) = \left(\frac{\theta}{n}\right) E(r) \tag{46}$$

is done in the same fashion as the one for the nonreplacement case (Epstein, 1954).

Suppose that at a certain time t, there have been exactly k failures, with observed total life V(t), where

$$V(t) = \sum_{i=1}^{k} x_{i,n} + (n-k)t \qquad 0 \leq k \leq r-1 \tag{47}$$

and where t is the elapsed time from time zero until the kth failure. Since V(t) is monotonically increasing, we should calculate V(t) at each failure and do one of three things

1. Continue the experiment as long as $V(t) < rC$ and $0 \leq k \leq r-1$.
2. Stop the experiment and accept H_o as soon as $V(t) > rC$ and $0 \leq k \leq r-1$.
3. Stop the experiment and reject H_o at the time $x_{r,n}$ if $V(t) < rC$.

Some properties of the above rules based on V(t) are presented. Define

$$x_{o,n} = 0 .$$

Then $$\sum_{i=1}^{r} x_{i,n} + (n-r)\, x_{r,n} \equiv \sum_{i=1}^{r} (n-i+1)(x_{i,n} - x_{i-1,n}) . \tag{48}$$

A new random variable is introduced as

$$\xi_i = (n-i+1)(x_{i,n} - x_{i-1,n}) \qquad i = 1, 2, \ldots, r \quad . \tag{49}$$

Therefore

$$\hat{\theta}_{r,n} > C$$

where $\hat{\theta}_{r,n}$ is given by (26) can now be written as

$$\sum_{i=1}^{r} \xi_i > rC \ . \tag{50}$$

All ξ_i are mutually independent random variables with common p. d.f. (Epstein, 1954)

$$\frac{1}{\theta} e^{-x/\theta} \qquad x > 0, \ \theta > 0 \ . \tag{51}$$

ξ_i is the time interval between the (i-1)st event and the ith event in a Poisson process with mean occurençe rate

$$\lambda = \frac{1}{\theta}$$

then

$$\sum_{i=1}^{r} \xi > rC$$

if and only if

$$0 \leq k \leq r-1 \ .$$

The probability of reaching a decision in exactly ρ=k failures is

$$\Pr(\rho = k|\theta) = p(k;\mu) \ , \qquad k = 0, 1, 2, \ldots, r_o-1 \tag{52}$$

$$\Pr(\rho = r|\theta) = 1 - \sum_{k=0}^{r-1} p(k;\mu) \tag{53}$$

where, for (52) and (53),

$$\mu = \frac{rC}{\theta}$$

and

$$p\ (k;\ \mu) = \mu^k\ e^{-\mu}/k! \quad . \tag{54}$$

Define

$$E(\rho) = \sum_{k=0}^{r} k\ Pr(\rho = k|\theta) = \mu \sum_{k=0}^{r-2} p(k;\ \mu) + r \left[1 - \sum_{k=0}^{r-1} p(k;\mu)\right] \tag{55}$$

$$E(T) = \sum_{k=1}^{r} Pr(\rho = k|\theta)\ E(x_{k,n}) \tag{56}$$

$$L(\theta) \quad \sum_{k=0}^{r-1} p(k;\ \mu) \quad . \tag{57}$$

It can be shown (Epstein, 1954) in the replacement case that the "best" region of acceptance when testing

$$H_o:\ \theta = \theta_o \quad \text{vs}$$

$$H_1:\ \theta = \theta_1 < \theta_o$$

based on the first r failures is

$$\hat{\theta}_{r,n} > C$$

with

$$\hat{\theta}_{r,n} = nx_{r,n}/r \ . \tag{58}$$

The region of acceptance for H_o is

$$x_{r,n} > C^*$$

$$= \frac{rC}{n}$$

Terminating the test at

$$\min(x_{r,n},\ C^*) \tag{59}$$

gives us the same test as before with

$$r = r_o \text{ and } C^* = T_o \quad .$$

A method for finding r_o and n when T_o is previously assigned is; r is the smallest integer so that

$$\frac{\chi^2(1-\alpha,2r)}{\chi^2(\beta,2r)} \geq \frac{\theta_o}{\theta_1} \qquad \theta_o > \theta_1 \quad . \tag{60}$$

Since

$$T_o = C^*$$

$$= \frac{rC}{n}$$

$$= \frac{\theta_o\ \chi^2(1-\alpha,2r)}{2n} \tag{61}$$

and

$$n = \left[\frac{\theta_o\ \chi^2(1-\alpha,2r)}{2T_o} \right] \quad . \tag{62}$$

In the nonreplacement case there is a method which is not as exact, but much easier than using binomial tables. Here, n and C are calcualted in the following manner,

$$n = \left[\frac{r_o}{1 - e^{T_o/C}} \right] \tag{63}$$

$$C = \frac{\theta_o \chi^2(1-\alpha, 2r_o)}{2r_o} \quad .$$

Let

$$\beta_{r_o,n} = \frac{1}{E(x_{r_o,n})} \quad . \tag{64}$$

Then a rule for accepting H_o could be to accept H_o when

$$\beta_{r_o,n}\, x_{r_o,n} > C \quad . \tag{65}$$

This rule gives an operating characteristic (O.C.) curve which is almost the same as the O.C. curve obtained from the rule of accepting H_o when

$$\hat{\theta}_{r_o,n} > C$$

where

$$T_o = C/\beta_{r_o,n} \quad . \tag{65}$$

When n is large,

$$\frac{1}{\beta_{r_o,n}} \sim \log \left[\frac{n}{n-r} \right] \quad . \tag{66}$$

An example of the testing procedure that has been covered is to find a replacement plan for T_o = 500 hours which will accept a lot with mean life = 10,000 hours at least 95% of the time and reject a lot with

mean life = 2,000 hours at least 95% of the time. Epstein (1954) computes $L(\theta)$, $E(T)$ and $E(r)$ at $\theta = 10{,}000$ and 2,000 hours, as

$$\theta_o = 10{,}000 \; , \quad \theta_1 = 2{,}000$$

$$\alpha = \beta = .05$$

$$r_o = \theta_o/\theta_1 = 5$$

$$\theta_o/T_o = 20$$

$$n = (1.97)(20) = 39 \quad .$$

Therefore the plan is to stop the experiment at

$$\min(x_{5,39}, 500) \quad .$$

The equation for $L(\theta)$ is given by

$$L(\theta) = 1 - \Pr\,(r = r_o|\theta) \; .$$

From Epstein (1954) for $\theta = 10{,}000$

$$\lambda T_o = 1.95$$

$$L(\theta) = .952$$

$$E(r) = 1.93$$

$$E(T) = 495$$

and for $\theta = 2{,}000$,

$$\lambda T_o = 9.75$$

$$L(\theta) = .034$$

$$E(r) = 4.95$$

$$E(T) = 254 \quad .$$

Referring to the same problem as above, Epstein (1954) again finds the values for the $L(\theta)$, $E(T)$, and $E(r)$ for the nonreplacement case. From the appropriate equations, the following results are obtained,

$$r_o = 5$$

$$n = 42 \quad .$$

For $\theta = 10{,}000$, the following results are calculated.

$$T_o/\theta = .05$$

$$p_o = 1 - e^{-T_o/\theta} = .049 \quad .$$

Using the tables from the National Bureau of Standards (1950), and the appropriate equations,

$$L(\theta) = .946$$

$$E(r) = 2.02$$

$$E(T) = 494$$

and for $\theta = 2{,}000$

$$L(\theta) = 0.031$$

$$E(r) = 4.91$$

$$E(T) = 248 \quad .$$

Epstein (1960c) gives two nonparametric tests for sequential life testing. The first test is set up on the basis of testing n items for a predetermined length of time t^*. At the end of t^* time we can say with 100 $(1-\alpha)$% confidence that at least 100b% of the population survives for length of time t^*, where

$$b = \left[1 + \frac{r+1}{n-1} \; F_\alpha(2r+2,\ 2n-r)\right]^{-1} \quad , \tag{67}$$

and $F_\alpha(r_1, r_2)$ is the value exceeded α% of the time in the F distribution. Where the underlying distribution is known to be exponential, a $100(1-\alpha)$% confidence interval for θ is given as

$$\theta > t^*\left[\log\left\{1 + \left(\frac{r+1}{n-r}\right) F_\alpha\ (2r+2,\ 2n-r)\right\}\right]^{-1} . \tag{68}$$

The second nonparametric test is designed in the following manner; items are drawn one at a time from the population and each item is tested separately until either the item fails or the length of time t^* has occurred. This is done until r items have failed. Again we have $100(1-\alpha)\%$ confidence that at least $100b\%$ of the population survives for a length of time t^*, where

$$b = \left[1 + \left(\frac{r}{n-r}\right) F_\alpha(2r,\ 2n-r)\right]^{-1} . \tag{69}$$

Again a one-tailed confidence interval of θ is given as

$$\theta > t^* \left[\log\left\{1 + \left(\frac{r}{n-r}\right) F_\alpha(2r,\ 2n-r)\right\}\right]^{-1} . \tag{70}$$

Confidence intervals have also been devised for the parametric estimation of θ (Epstein, 1960c). A two-tailed confidence interval on θ is given as follows;

$$\frac{2T}{\chi^2(\alpha/2, 2r+2)} < \theta < \frac{2T}{\chi^2(1-\alpha/2, 2r)} \tag{71}$$

where r = number of failures up to time T. A one-tailed confidence interval on θ is given as

$$\theta > \frac{2T}{\chi^2(\alpha, 2r+2)} . \tag{72}$$

Another method for testing hypotheses is by regular sequential analysis methods which are discussed by Epstein and Sobel (1955). The maximum likelihood estimate for this procedure is based on the following

probability ratio

$$\frac{P_1(x_{(1)}x_{(2)} \cdot \cdot \cdot , x_{1r}, t; \theta_1)}{P_1(x_{(1)}x_{(2)} \cdot \cdot \cdot , x_{1r}, t; \theta_0)} = \prod_{i=1}^{r} \frac{f(x_{1i}, \theta_1)}{f(x_{1i}, \theta_0)} \left[\frac{1 - F(t, \theta_1)}{1 - F(t, \theta_0)}\right]^{n_1 - r_1} . \quad (73)$$

This ratio is used for the nonreplacement case, and if the ratio is between A and B, where

$$B = \frac{\beta}{1-\alpha}$$

and

$$A = \frac{1-\beta}{\alpha}$$

a decision is not made from the first sample, and additional samples are drawn until a decision is made. (73) may be set up in another manner so that both replacement and nonreplacement cases may be handled.

$$B < \left(\frac{\theta_0}{\theta_1}\right)^r \exp\left[-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) V(t)\right] < A \quad (74)$$

In the nonreplacement case

$$V(t) = \sum_{i=1}^{r} (n-i+1)(x_i - x_{i-1}) + (n-r)(t-x_r)$$

$$= \sum_{i=1}^{r} x_i + (n-r)(t-x_r) \quad (75)$$

and for the replacement case

$$V(t) = nt \ . \quad (76)$$

Any time the inequality of (74) is violated, the experiment is stopped. If the value is less than B, H_o is rejected, and if it is greater than A, H_o is accepted.

According to Wald (1947), in order to have a test with exactly the strength (α, θ), A and B should be replaced by $A^* \leq$ and $B^* \geq B$. Formulae for computing these values plus O. C. and average sample number curves are given in Burman (1946) and Dvoretsky, Kiefer and Wolfowitz (1953).

If we wish to graph the data continuously, (74) can be written as

$$- h_o + rs < V(t) < h_o + rs \tag{77}$$

where

$$h_o = \frac{-\log B}{1/\theta_1 - 1/\theta_o}$$

$$h_1 = \frac{\log A}{1/\theta_1 - 1/\theta_o}$$

and

$$s = \frac{\log (\theta_o/\theta_1)}{1/\theta_1 - 1/\theta_o} \quad .$$

The o.c. curve which represents the probability of accepting H_o when θ is the true parameter is given approximately by the following two parametric equations

$$L(\theta) = \frac{A^h - 1}{A^h - B^h}$$

$$\theta = \frac{(\theta_o/\theta_1)^h - 1}{h(1/\theta_1 - 1/\theta_o)}$$

letting h run through all real values. It turns out that if five particular values of $L(\theta)$ are found,

$$\theta = 0,\ \theta_1,\ s,\ \theta_o,\ \text{and } \infty$$

the entire curve can be sketched.

The following terms are given by Epstein and Sobel (1955),

$$E(r) \sim \begin{cases} \dfrac{L(\theta)\ \log B + \left[1 - L(\theta)\right]\ \log A}{\log\left[\theta_o/\theta_1\right] - \theta(1/\theta_1 - 1/\theta_o)} = \dfrac{h_1 - L(\theta)(h_o + h_1)}{s - \theta} & \theta \neq s \\[2ex] \dfrac{-\log A \log B}{\log\ (\theta_o/\theta_1)^{\ 2}} = \dfrac{h_o h_1}{s^2} & \theta = s\ . \end{cases} \tag{80}$$

If

$$k = \theta_o/\theta_1$$

and

$$\theta = \theta_1,\ s,\ \text{or } \theta_o \tag{81}$$

then

$$E_{\theta_1}(r) \sim \frac{\left[\beta\ \log B + (1-\beta)\ \log A\right]}{\left[\log k - (k-1)/k\right]}$$

$$E_s(r) \sim \frac{-\log A \log B}{(\log k)^2} \tag{82}$$

$$E_{\theta_o}(r) \sim \frac{\left[(1-\alpha)\ \log B + \alpha \log A\right]}{\left[\log k - (k-1)/k\right]}\ . \tag{83}$$

An example of hypothesis testing using the above method is to find a sequential replacement procedure for testing

$$H_o: \ \theta = \theta_o = 7500 \text{ hours} \quad \text{vs}$$

$$H_1: \ \theta = \theta_1 = 2500 \text{ hours}$$

$$\alpha = \beta = .05$$

$$n = 100 .$$

An approximate solution is

$$\frac{1}{19} < 3^r \, e^{-V(t)/3750} < 19$$

with

$$V(t) = 100t .$$

For this test,

$$\alpha' = .032 \text{ and } \beta' = .051.$$

In the case being considered, $B = B^*$, since the acceptance of H_o involves no excess over the boundary. The exact solution (B, A^*) is

$$\frac{1}{19} < 3^r \, e^{-100t/3750} < 13.25$$

$$\alpha = \beta = .05 .$$

Now find $E(r)$ and $E(T)$ for

$$\theta = 0, \ \theta_1(= 2500), \ S(= + 115), \ \theta_o(= 7500), \text{ and } \infty$$

$$E(r) = \left\{ \begin{array}{llll} \theta = & \theta_o & S & \theta_1 \\ & 2.97 & 7.22 & 6.21 \\ & 3.03 & 8.10 & 7.00 \end{array} \right. \quad \begin{array}{l} \\ \text{using } (B, A) \text{ rule} \\ \text{using } (B, A^*) \text{ rule} \end{array}$$

$$E(T) = \left\{ \begin{array}{lllll} \theta = & 0 & \theta_1 & S & \theta_o \\ & 0 & 175 & 333 & 227 \\ & 0 & 155 & 297 & 220 \end{array} \right. \quad \begin{array}{l} \\ \text{using } (B, A) \text{ rule} \\ \text{using } (B, A^*) \text{ rule} \end{array}$$

$$\theta = \infty \quad \text{for } e^{-100t^{\infty}/3750} = \frac{1}{19}$$

$$t_{\infty} = E_{\infty}(t) = 110 \quad .$$

In terms of B, n, θ_o, and k

$$t_{\infty} = \frac{-\theta_o \log B}{n(k-1)} \quad .$$

If no items fail by t_{∞}, we stop the experiment and accept H_o .

CHAPTER III

ESTIMATION OF PARAMETERS

Now assume that the mean life of our population is unknown but the population is distributed exponentially. The purpose of the life test now is to estimate the mean life after observing a certain number of failures or at some specific time.

Bartholomew (1957) looks at a true industrial situation. Suppose that we have records of the installation dates of certain pieces of equipment. Some of the equipment has failed and has not been replaced, and others are still in use.

In this particular experiment, the experimenter does not have complete control over the life test. The equipment has not all been installed at the same time, and he may be required to estimate its mean life at any time, no matter what stage the experiment is in.

The time which has elapsed since the installment of a certain piece of equipment is given as T_i, and the life length is given as t_i. This is known only if

$$t_i \leq T_i \; .$$

The only thing which is exactly controlled by the experimenter is the size of the sample. It is assumed that each item has the same life distribution with a given density $p(t)$. The probability that an item has failed at time T_i is

$$P_i = \int_0^{T_i} p(t)\, dt \qquad Q_i = 1 - P_i \quad . \tag{84}$$

Our estimate will be obtained by the maximum likelihood method which is given by

$$l^* = \prod_{i=1}^{n} P_i^{a_i}\, Q_i^{1-a_i} \left[p(t_i) \,/\, P_i \right]^{a_i} \tag{85}$$

$$a_i = 1 \quad \text{if item has failed}$$

$$a_i = 0 \quad \text{if item has not failed}$$

$$k = \sum_{i=1}^{n} a_i$$

$$L = \log l^* = \sum_{i=1}^{n} \left[(1 - a_i) \log Q_i + a_i \log p(t_i) \right] \quad . \tag{86}$$

Since we are assuming all distributions to be exponential, an explicit solution can be obtained,

$$p(t) = 1/\theta\, e^{-t/\theta} \qquad 0 \leq t < \infty, \quad \theta > 0 \tag{87}$$

$$L = - \sum_{i=1}^{n} \left[(1 - a_i)^{T_i/\theta} + a_i \, (\log \theta + t_i/\theta) \right] \tag{88}$$

setting $\frac{\delta L}{\delta \theta} = 0$ we get

$$\sum_{i=1}^{n} \left[(1 - a_i)\, T_i/\hat{\theta}^2 - a_i/\hat{\theta} + a_i t_i/\hat{\theta}^2 \right] = 0 \tag{89}$$

$$\hat{\theta} = 1/k \sum_{i=1}^{n} \left[a_i t_i + (1 - a_i)\, T_i \right] \quad . \tag{90}$$

The variance for large n has been given by R. A. Fisher

$$\mathrm{Var}\,(\hat{\theta}) = -\frac{1}{E\left(\frac{\delta^2 L}{\delta\theta^2}\right)} \tag{91}$$

$$\frac{\delta^2 L}{\delta^2\theta} = \frac{k}{\theta^2} - \frac{2k\hat{\theta}}{\theta^3} \quad . \tag{92}$$

We will show later that

$$E(k,\, \hat{\theta}) = \theta \sum_{i=1}^{n} P_i \quad . \tag{93}$$

Since k comes from binomial sampling with unequal probabilities, P_i, the following is obtained,

$$E(k) = \sum_{i=1}^{n} P_i \quad . \tag{94}$$

From these results,

$$E\left(\frac{\delta^2 L}{\delta\theta^2}\right) = \frac{\sum_{i=1}^{n} P_i}{\theta^2} \quad . \tag{95}$$

Therefore in large samples

$$\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{\sum_{i=1}^{n} P_i} \quad . \tag{96}$$

Since θ is unknown, we can estimate this by

$$\mathrm{Var}\,(\hat{\theta}) = \frac{\hat{\theta}^2}{\sum_{i=1}^{n} P_i} \quad . \tag{97}$$

Because of (94), a quick estimate of the variance would be

$$\text{Var}\ (\hat{\theta}) = \frac{\hat{\theta}^2}{k} \quad . \tag{98}$$

Since the maximum likelihood estimator is often biased in small samples, let's investigate the bias of $\hat{\theta}$;

$$\text{Cov}\ (k, \hat{\theta}) = E(k; \hat{\theta}) - E(k) \cdot E(\hat{\theta})$$

$$E(\hat{\theta}) = \frac{E(k; \hat{\theta})}{E(k)} - \frac{\text{Cov}\ (k, \hat{\theta})}{E(k)} \tag{99}$$

$k\hat{\theta}$ can be regarded as the sum of n variates from the population. Therefore

$$p(x_i) = \frac{1}{\theta}\ e^{-x_i/\theta} \qquad 0 \le x_i < T_i$$

with a probability of

$$e^{-T_i/\theta} \qquad\qquad x_i = T_i \quad .$$

The expected value of the ith member of the sum can be shown to be

$$(\theta P_i - Q_i T_i) + Q_i T_i \ .$$

Therefore

$$E(k; \hat{\theta}) = \theta \sum_{i=1}^{n} P_i \ . \tag{100}$$

Using (94),

$$E(\hat{\theta}) = \theta - \frac{\text{Cov}\ (k, \hat{\theta})}{\sum_{i=1}^{n} P_i} \quad . \tag{101}$$

The bias of our estimate is

$$\left| \frac{\mathrm{Cov}\ (k, \hat{\theta})}{\sum_{i=1}^{n} P_i} \right|$$

$$\left| \mathrm{Cov}\ (k, \hat{\theta}) \right| \leq [\mathrm{Var}\ (k)\ \mathrm{Var}(\hat{\theta})]^{1/2}$$

$$= [\sum_{i=1}^{n} P_i Q_i]^{1/2} \qquad (102)$$

$$\leq E(\hat{\theta})\ .$$

Therefore

$$|E(\hat{\theta}) - \theta| \leq [\sum_{i=1}^{n} P_i Q_i]^{1/2}$$

$$\leq \frac{E(\hat{\theta})}{\sum_{i=1}^{n} P_i}$$

since

$$\sum_{i=1}^{n} P_i Q_i \leq \sum_{i=1}^{n} P_i\ . \qquad (103)$$

Also by using the limiting value of the standard error of θ, we obtain a weaker result of our bias

$$|E(\hat{\theta}) - \theta| \leq \frac{S.E.(\hat{\theta})}{\sum_{i=1}^{n} P_i}$$

which is approximately equal to

$$\frac{\theta}{\sum_{i=1}^{n} P_i} \quad . \qquad (104)$$

Exact values of $E(\hat{\theta})$ may be obtained from tables by Grab and Savage (1954).

Let's look at an example of the previous discussion. A random sample of size ten is drawn from a population of pieces of equipment. The data is recorded in the following table.

Results of a life test on Ten Pieces of Equipment

Item Number	1	2	3	4	5	6	7	8	9	10
Dates of Instl.	11 June	21 June	22 June	2 July	21 July	31 July	31 July	1 Aug.	2 Aug.	10 Aug.
Dates of Fail.	13 June	-	12 Aug.	-	23 Aug.	27 Aug.	14 Aug.	25 Aug.	6 Aug.	-
Life in days	2	(119)	51	(77)	33	27	14	24	4	(37)
T_i	81	72	70	60	41	31	31	30	29	21

The estimate of the mean life is asked for on August 31.

$$\hat{\theta} = \frac{1}{7}(2 + 51 + 33 + 27 + 14 + 24 + 4 + 72 + 60 + 21)$$

$$= 44 \text{ days}$$

$$\text{std. dev.} \tilde{=} \hat{\theta} / \sqrt{\sum_{i=1}^{n} P_i}$$

$$= \frac{44}{\sqrt{6.15}} = 17.7 \quad .$$

Using all ten values instead of seven,

$$\hat{\theta} = 38.8 \quad .$$

We tend to overestimate θ by $\hat{\theta}$, as illustrated in the example where $\hat{\theta} = 44$. This should be intuitively true since we are using the life length of all n items up to T_i, whether the item has failed or not. All n values are added together and divided by k. The true value of θ in the example is 40.8 .

Another type of problem which is encountered by engineers (Mendenhall and Harder, 1958) is where failure is caused by more than one event. Suppose we are studying the failure of electron tubes, and we want to determine the portion of tubes which fail from gaseous defects or mechanical defects from normal deterioration of the cathode. It would also be desirable to know the distribution of failure for each cause. We are now studying a population with $s = 2$ subpopulations, representing failure types, mixed in proportions p and q where

$$q = 1 - p \qquad 0 \leq p \leq 1 \quad .$$

Let's assume that the failure times for the ith subpopulation, $i = 1, 2$, have a c.d.f.

$$F_i(t) = 1 - e^{-t/\alpha_i} \qquad 0 \leq t < \infty \; . \tag{105}$$

If p is the proportion of units in subpopulation 1, then the c.d.f. for the population is

$$F(t) = pF_1(t) + qF_2(t) \tag{106}$$

and the p.d.f. is

$$f(t) = pf_1(t) + qf_2(t) \; . \tag{107}$$

Define

$$G_i(t) = 1 - F_i(t) \tag{108}$$

$$G(t) = 1 - F(t) \; . \tag{109}$$

$G(t)$ is called the survival function or the probability that a unit will survive until time t .

If all items of the population were put on test, the proportion in each subpopulation would change from time to time. At time t they would be in the proportions

$$p(t) : 1 - p(t) .$$

Define these two proportions as conditional mixture proportions

$$p(t) = \frac{p\ G_1(t)}{G(t)} \qquad p(0) = p . \tag{110}$$

Testing n items from the population, the test will be terminated at the end of a predetermined time T. At this time there will be r failures, r_1 from subpopulation 1 and r_2 from subpopulation 2 . If t_{ij} is the failure time of the jth item in the ith subpopulation, $j = 1, 2, \ldots, r_i$, then let

$$x = t/T \tag{111}$$

and

$$\beta_i = \alpha_i/T \tag{112}$$

now

$$F_i(x) = 1 - e^{-x/\beta_i} \qquad 0 \leq x < \infty . \tag{113}$$

The probability of r_1 and r_2 failures along with (n-r) survivals is multinomial.

$$P_r(r_1, r_2, n-r|n) = \frac{n!}{r_1! r_2! (n-r)!} \left[pF_1(1)\right]^{r_1} \left[qF_2(1)\right]^{r_2} \left[G(1)\right]^{n-r} . \tag{114}$$

From the ordered observations, $x_{i1}, x_{i2}, \ldots, x_{ir_i}$; given r_i and $x_{ij} \leq 1$ is

$$\Pr(x_{i1}, x_{i2}, \ldots, x_{ir_i} | r_i ; x_{ij} \leq 1) = \frac{r_i! \prod_{j=1}^{r_i} f_i(x_{ij})}{[F_i(1)]^{r_i}} . \quad (115)$$

From this it follows that the likelihood 1^* is

$$1^* = \frac{n!}{(n-r)!} G(1)^{n-r} p^{r_1} q^{r_2} \prod_{j=1}^{r_1} f_1(x_{1j}) \prod_{j=1}^{r_2} f_2(x_{2j}) . \quad (116)$$

Taking the first partial derivatives of $\log 1^* = L$

$$\frac{\delta L}{\delta \beta_2} = \frac{k(n-r)}{\beta_2^2} - \frac{r_1}{\beta_1} + \frac{r_1 \bar{x}_1}{\beta_1^2} \quad (117)$$

$$\frac{\delta L}{\delta \beta_2} = \frac{(1-k)(n-r)}{\beta_2^2} - \frac{r_2}{\beta_2} + \frac{r_2 \bar{x}_2}{\beta_2^2} \quad (118)$$

$$\frac{\delta L}{\delta p} = \frac{k(n-r) + r_1}{p} - \frac{(1-k)(n-r) + r_2}{q} \quad (119)$$

where (Mendenhall and Harder, 1958)

$$k = \frac{p\, e^{-1/\beta_1}}{p\, e^{-1/\beta_1} + q\, e^{(1/\beta_1 - 1/\beta_2)}}$$

$$= \frac{1}{1 + (q/p)^{(1/\beta_1 - 1/\beta_2)}} \qquad p(1) = k . \quad (120)$$

Set the partials equal to zero, and, since the test has been terminated, $x = 1$, to get

$$\hat{p} = r_1/n + \hat{k}\,(n-r)/n \quad (121)$$

$$\hat{\beta}_1 = \bar{x}_1 + \hat{k}\,(n-r)/r_1 \quad (122)$$

$$\hat{\beta}_2 = \bar{x}_2 + (1-\hat{k})(n-r)/r_2 \qquad (123)$$

$$\bar{x}_i = \sum_{j=1}^{r_i} x_{ij}/r_i \qquad i = 1,\ 2 \qquad (124)$$

$$\hat{k} = \frac{1}{1 + (\hat{q}/\hat{p}) \exp \dfrac{1}{\beta_1} = \dfrac{1}{\beta_2}} \quad . \qquad (125)$$

Solving (121), (122), (123), and (125) simultaneously, we get

$$\hat{k} = g(\hat{k}) \qquad 0 \leq \hat{k} \leq 1 \quad .$$

A good first approximation to $\hat{k}$ can be obtained by using a modification of the maximum likelihood estimate obtained by Deemer and Votaw (1952).

The maximum likelihood estimate of β_i, where the experiment has been stopped at time T, comes from the solution of

$$(\beta_i - \bar{x}_i)\ (e^{1/\beta_i} - 1) = 1 \ . \qquad (126)$$

$\hat{\beta}$ can be obtained from figure 1 in Mendenhall and Harder (1958), p 507 .

In practical cases it is reasonable to say that β_i is either very large or $p = 0$ when $r_i = 0$. Let's say that β_i is very large when $r_i = 0$. But in experiments, both n and T should be large enough so that the probability of $r_1 = 0$ or $r_2 = 0$ is very small.

Assume that in our example we know, for some non-statistical reason, that

$$\beta_1 \leq \beta_2$$

but the procedure we have been describing produces estimates where

$$\hat{\beta}_1 > \hat{\beta}_2 .$$

So if

$$\hat{\beta}_1 > \hat{\beta}_2$$

given

$$\beta_1 \leq \beta_2$$

we say that a crossover has occured. It is now reasonable to say that

$$\hat{\beta}_1 = \hat{\beta}_2$$

when a crossover occurs. The maximum likelihood estimate of

$$\beta_1 = \beta_2 = \beta$$

is given as

$$\hat{\beta} = \frac{r_1\bar{x}_1 + r_2\bar{x}_2 + (n-r)}{r} \tag{127}$$

and

$$\hat{p} = \frac{r_1}{n} \quad . \tag{128}$$

As an example of multiple failure classification, let's look at some work which was done by Acheson and McEwlee (1951).

The data recorded in the tables (Mendenhall and Harder, 1958:508) refer to failure times of ARC- 1 VHF communication transmitter-recievers of a single commercial airline. The data in the first table is confirmed failures and the data in the second table is unconfirmed failures. An unconfirmed

failure is defined as the unit being reported to have failed and taken from the airplane, but after further testing it is recorded as still working properly. Each unit is automatically removed from the plane after 630 hours, so 630 is time T. It is desirable to estimate the proportion of unconfirmed failures in the population.

$$n = 369,\ r_1 = 107,\ r_2 = 218,\ r = 325$$

$$\bar{x}_1 = \frac{t_1}{T} = 0.3034862$$

$$\bar{x}_2 = 0.3644677$$

$$\hat{\beta}_1 = 0.3035 + 0.4112\hat{k}$$

$$\hat{\beta}_2 = 0.5663 - 0.2018\hat{k}$$

$$\hat{p} = 0.2900 + 0.1192\hat{k}\ .$$

We can simplify the iterative solution by using the following table.

The first step is to estimate β from figure 1 in Mendenhall and Harder (1958). For $\bar{x}_1 = 0.303$, the first estimate of β_1 is $\hat{\beta}_{10} = 0.380$. The corresponding estimate of k is $\hat{k}_o = 0.186$, which is obtained by solving for $\hat{k}$ in

$$\hat{\beta}_1 = 0.3035 + 0.4112\hat{k}\ .$$

By using this value of $\hat{k}$, we can quickly find $\hat{\beta}_{20}$ and $\hat{p}_o$ from the above equation. These values are in row u = 0

u	$\hat{k}_u$	$\hat{\beta}_{1u}$	$\hat{\beta}_{2u}$	p_u	v_u	$g(\hat{k}_u)$	D_u
0	0.186	0.380	0.529	0.312	4.622	0.1779	-0.0081
1	0.166	0.3718	0.5328	0.3098	5.024	0.1660	0.0000

Next we compute

$$g(\hat{k}_o) = \frac{1}{1 + (\hat{q}_o/\hat{p}_o)\,\exp\left[\frac{1}{\hat{\beta}_{10}} - \frac{1}{\hat{\beta}_{20}}\right]}$$

$$D_o = g(\hat{k}_o) - \hat{k}_o \quad .$$

The value of $\hat{k}$ which occurs when D = 0 is the one which corresponds to the solution of the maximum likelihood estimate. Since D is positive or zero for $\hat{k} = 0$

$$0 < \hat{k} < 0.186 \; .$$

The change in $\hat{k}$, $d(\hat{k}_o)$, can be computed by the following formula

$$d(\hat{k}_o) = \frac{D_o}{1 + g\,(\hat{k}_o)^2\,(dv_o/dk_o)}$$

$$= \frac{(-0.0081)}{1 + (0.1779)^2\,(-19.04)}$$

$$= -\,0.02$$

$$k_1 = 0.166$$

Since $D_1 = 0.0000$, we now have the maximum likelihood estimates of the parameters. The proportion of unconfirmed failures is

$$\hat{p}_1 = 0.3098$$

Going back to Chapter I, from Theorem 1, we again give the estimates of θ for cases 1, 2, and 3. For cases 1, 2, and 3 respectively, $\hat{\theta}_i (i=1,2,3)$ is given as follows

$$\hat{\theta}_1 = \frac{\sum_{j=1}^{k} V_j}{R} \quad . \tag{129}$$

The V_j's are calculated according to (15) .

$$\hat{\theta}_2 = \frac{\sum_{j=1}^{k} V_j^*}{R} \quad . \tag{130}$$

where the V_j^*'s come from (17) .

$$\hat{\theta}_3 = \frac{\sum_{j=1}^{k} V_j'}{R} \tag{131}$$

where the V_j''s come from (16) .

Let's now look at a structure which usually requires more than one failure within the structure before the entire structure fails. This type of problem was considered by Birnbaum and Saunders (1958) .

If it takes k failures for the structure to fail, then define

S_k = life length until k failures have occurred.

δ = instantaneous damage on a component of the structure.

with (17) and (18) being the c.d.f. and p.d.f. respectively.

Referring to (19), let

$$U = 2 \int_0^{S_k} \gamma_\delta (t)\, dt \tag{132}$$

Then U is a r. v. with density

$$h(U;\,k) = \frac{1}{(k-1)!2^k}\, U^{k-1}\, e^{-U/2} \qquad U > 0 \tag{133}$$

which is a $\chi^2(2k)$.

If θ is estimated by (130), this is said to be the "best" estimate in the sense that it is unbiased, minimum variance, efficient and sufficient (Epstein, 1956). The variance of $\hat{\theta}$ is given by

$$\text{Var }(\hat{\theta}) = \frac{\hat{\theta}}{r} \;.$$

A two-tailed confidence interval for θ is given as follows

$$\frac{2r\ \hat{\theta}_{r,n}}{\chi^2(\alpha/2, 2r)} < \theta < \frac{2r\ \hat{\theta}_{r,n}}{\chi^2(1 - \alpha/2,\ 2r)} \;. \tag{134}$$

The final estimating procedure is from Plackett (1959). Suppose that we are going to burn 10 lamps for 2 months, and on the basis of the number of failures in two months, we would like to predict the percentage of lamps which will burn more than 6 months.

The underlying distribution is assumed to be normal, with mean θ and standard deviation σ .

At the end of 2 months, 7 lamps have failed, and their respective life lengths are: 1050, 1089, 1272, 1302, 1345, 1380, and 1423 hours.

The first method of estimation is graphical (Plackett, 1959). From this figure, it is possible to estimate θ and σ by using the end points

$$\hat{\theta} - 1.54\hat{\sigma} = 1050$$

$$\hat{\theta} + 0.38\hat{\sigma} = 1423$$

$$\hat{\theta} = 1349$$

$$\hat{\sigma} = 194$$

The efficiency of these estimates compared to the "best" estimate is 89.2% for $\hat{\theta}$ and 89.7% for $\hat{\sigma}$.

Another method for estimating θ and σ is the following. To estimate θ, multiply each value by 0.0244, 0.0636, 0.0818, 0.0961, 0.1089, 0.1207, and 0.5045 (Gupta, 1958) respectively and add the products.

$$\hat{\theta} = 1355$$

In the same manner, multiply each value by -0.3253, -0.1757, -0.1058, -0.0502, -0.0007, 0.0470, and 0.6106 .

$$\hat{\sigma} = 200$$

The appropriate coefficients for samples of size $n \leq 10$ have been tabled by Gupta (1958). For large samples ($n > 50$) (Plackett, 1959),

$$p_i = \frac{\text{average rank}}{n}$$

t_i = corresponding normal equivalent deviations.

For an example, let $n = 300$, sampling from the same population as above.

Central Life y_i	f	Cum. f	Average rank	p_i	t_i
975	2	2	1.5	0.0050	-2.576
1025	2	4	3.5	0.0117	-2.267
1075	3	7	6.0	0.0200	-2.054
1125	6	13	10.5	0.0350	-1.812
1175	7	20	17.0	0.0567	-1.583
1225	12	32	26.5	0.0883	-1.351
1275	16	48	40.5	0.1350	-1.103
1325	20	68	58.5	0.1950	-0.860
1375	24	92	80.5	0.2683	-0.618
1425	27	119	106.0	0.3533	-0.376

A method which uses the data in the table, is given by Plackett (1959), giving the following results;

$$\hat{\theta} = 1503 \quad \text{and} \quad \hat{\sigma} = 207 \quad .$$

The approximate method using only the end points, gives these results

$$\hat{\theta} - 2.576\,\hat{\sigma} = 975$$

$$\hat{\theta} - 0.376\,\hat{\sigma} = 1425$$

$$\hat{\theta} = 1502 \text{ and } \hat{\sigma} = 205 \quad .$$

The estimates obtained by solving awkward equations for the maximum likelihood estimate gave

$$\hat{\theta} = 1503 \quad \text{and} \quad \hat{\sigma} = 207 \quad .$$

These results are not much different from the ones using the easier method, using only the end points.

CONCLUSION

There may be a good answer to the question concerning the validity of the exponential distribution being used in life testing situations. There are those who favor using the exponential distribution in life testing situations and those who are opposed. The answer may be that in a sense, they are both right. For example, suppose the data appears to be non-exponential due to early failures (Miller, 1960). Let's assume that the failure rate of whatever is being tested is known to follow the exponential distribution. A random sample is drawn from this population and placed on test. For some unknown reason there are many failures early in the experiment. Because of these early failures, it is possible to show that the data does not fit the exponential distribution, but on the other hand, after these early failures, the other failures follow the exponential very closely. So if this were an actual case, both sides would have a good argument.

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SEQUENTIAL LIFE TESTING

by

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AN ABSTRACT OF A MASTER'S REPORT

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ABSTRACT

The purpose of this paper is to show some aspects of sequential life testing and their applications.

Life testing is used where the data being studied is the length of life of the product, machine etc., which discontinues because of failure. There are several types of failure. An item can fail through normal constant wear, fail only when a force is exerted on the item, or fail because of an accident which damages or destroys the item.

The first chapter of the report gives the underlying distribution of sequential life testing and some related theorems. The distribution most often used is the exponential distribution. Both the pros and cons of using the exponential are given. The distribution really depends on what is being studied, but some authorities tend to use the exponential nearly all the time. Some of the other popular distributions are the Weibull, Poisson, and gamma. Although it is not mentioned in the first chapter, the distribution of one method discussed in Chapter III, is assumed to be normal.

Chapter II is concerned with testing hypotheses. The most frequent type of hypothesis which is tested is

$$H_o: \theta = \theta_o \quad \text{vs}$$

$$H_1: \theta = \theta_1 < \theta_o$$

The first method is to determine the sample size, a number of failures r, $r \leq n$, and a time T. If r failures occur before time T, then H_o is rejected, and if time T occurs and there have only been k failures, $k < r$, then H_o is accepted.

Two nonparametric tests are given for determining the percentage of items which will live a certain length of time.

Another method of testing hypotheses is by using the technique of regular sequential analysis. This involves the evaluation of a function inside inequality signs, after each failure. Whenever the inequality is violated, the testing is stopped, and a decision is made concerning the null hypothesis.

Chapter III of the report deals with estimation of parameters in life testing situations. A typical problem would be that a company has developed a new product, and the company officials would like to know the mean life of the product.

One method of estimation is to use a design which is similar to one used in hypotheses testing. From a sample of the population, test the product until r failures occur. Then from the information recorded, estimate the mean life of the product.

There are various uncontrolled factors which may occur in life testing situations, these factors should be taken into consideration while making estimates.

Some methods of estimation are more exact, theoretically speaking, than others. But in practice the approximate methods, which are easier to calculate, seem to give almost as good results.